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# On Oscillation of Solutions of Even Order Nonlinear Differential Equations

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In this paper we are dealing with differential equations of the form:

$$x^{(2n)} + p(t)\, g(x, x^{(1)}, \ldots, x^{(2n-1)}) = 0. \tag{*}$$

Ličko and Švec, generalizing previous results of Atkinton [*1*], and Belohorec [*2*], proved in [*5*] that in the case $g = g(x) = x^{\alpha}$, $\alpha > 1$, $\alpha = q/r$ ($q, r$ odd positive integers relatively prime), $p \in \mathbf{C}[t_0, +\infty)$, $p(t) > 0$, a necessary and sufficient condition for all solutions of (*) to oscillate is

$$\int_{t_0}^{+\infty} t^{2n-1} p(t)\, dt = +\infty. \tag{A}$$

Here we give two theorems. The first one shows that (A) is sufficient for all bounded solutions of (*) to oscillate, and the second gives a sufficient condition for all solutions of (*) to oscillate provided that $g = g(x)$.

Throughout the paper we consider only solutions of (*) which are nontrivial and valid for all large $t$.

1. THEOREM 1. *Consider* (*) *under the following assumptions*:

(i) $p : I \to \mathbb{R}_+ = (0, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geqslant 0$, $p \in \mathbf{C}[t_0, +\infty)$, and (A) is satisfied

(ii) $g : \mathbb{R}^n \to \mathbb{R} = (-\infty, +\infty)$, $x_1 g(x_1, x_2, \ldots, x_{2n}) > 0$ *for* $x_1 \neq 0$, *and continuous on* $\mathbb{R}^n$;

*then, under the above conditions, every bounded solution of* (*) *is oscillatory.*

*Proof.* The proof is by contradiction. Suppose that the conclusion is not true. Then there exists a bounded solution $x(t)$ of (*) which is non-

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oscillatory on $I$. We may (and do) assume that $x(t)$ is defined and positive on $I$. In this case, Eq. (*) gives

$$x^{(2n)} = -p(t)\, g(x(t), x^{(1)}(t),..., x^{(2n-1)}(t)) < 0 \tag{1}$$

for every $t \in I$. Thus, $x^{(2n-1)}(t)$ is decreasing on $I$. Now, $x^{(2n-1)}(t)$ must be positive for large $t$. In fact, if $x^{(2n-1)}(t)$ was eventually negative on $I$, then this would imply that $x(t)$ is eventually negative on $I$, a contradiction. Proceeding in the same way, we can show that $(-1)^m x^{(2n-m)}(t) < 0$ for every $m = 1, 2,..., 2n - 1$ and for sufficiently large $t$. Let $t_1$ be such that the previous inequality holds for every $t \geqslant t_1$, and consider the function

$$F(t) = t^{2n-1} x^{(2n-1)}(t), \qquad t \in [t_1, +\infty). \tag{2}$$

By differentiation of $F$ we obtain

$$F'(t) = -t^{2n-1} p(t)\, g(x(t), x^{(1)}(t),..., x^{(2n-1)}(t)) + (2n-1) t^{2(n-1)} x^{(2n-1)}(t). \tag{3}$$

Now, it is easy to show that $\lim_{t\to+\infty} g(x(t), x^{(1)}(t),..., x^{(2n-1)}(t))$ exists and is finite and positive. In fact, this is an immediate consequence of the continuity of $g$, the monotonicity of the derivatives of $x(t)$ up to the order $2n - 1$, and the fact that none of these derivatives can tend to a nonzero limit, for $x(t)$ is bounded on $I$. So, if $\epsilon > 0$ is such that $\epsilon < g(a_0, 0,..., 0) = A$, where $a_0 = \lim_{t\to+\infty} x(t)$, then there exists a $t_2 \geqslant t_1$ with the property

$$A - \epsilon < g(x(t), x^{(1)}(t),..., x^{(2n-1)}(t)) < A + \epsilon \tag{4}$$

for every $t \geqslant t_2$.

Thus, by use of (4) and integration of (3) from $t_2$ to $t \geqslant t_2$ we get

$$\begin{aligned} F(t) - F(t_2) \leqslant -(A - \epsilon) &\int_{t_2}^{t} s^{2n-1} p(s)\, ds \\ &+ (2n-1) \int_{t_2}^{t} s^{2(n-1)} x^{(2n-1)}(s)\, ds, \end{aligned} \tag{5}$$

which, since $F$ is positive, implies that

$$\lim_{t\to+\infty} \int_{t_2}^{t} s^{2(n-1)} x^{(2n-1)}(s)\, ds = +\infty. \tag{6}$$

On the other hand, computing the above integral we find

$$\int_{t_2}^{t} s^{2(n-1)} x^{(2n-1)}(s)\, ds = s^{2(n-1)} x^{(2n-2)}(s)\Big]_{t_2}^{t} - 2(n-1) \int_{t_2}^{t} s^{2n-3} x^{(2n-2)}(s)\, ds, \tag{7}$$

from which, by taking into account (6), we obtain

$$\lim_{t\to+\infty} \int_{t_2}^{t} s^{2n-3}x^{(2n-2)}(s)\,ds = -\infty, \tag{8}$$

and similarly by induction

$$\begin{aligned} &\int_{t_2}^{+\infty} t^{2n-(m+1)}x^{(2n-m)}(t)\,dt = +\infty \qquad \text{if} \quad m = 1, 3, \ldots, 2n-1 \\ &\int_{t_2}^{+\infty} t^{2n-(m+1)}x^{(2n-m)}(t)\,dt = -\infty \qquad \text{if} \quad m = 2, 4, \ldots, 2(n-1) \end{aligned} \tag{9}$$

Now, the second of (9) for $m = 2(n-1)$ gives

$$\int_{t_2}^{+\infty} tx''(t)\,dt = -\infty, \tag{10}$$

or

$$\lim_{t\to+\infty} [tx'(t) - t_2x'(t_2) + x(t_2) - x(t)] = -\infty \tag{11}$$

a contradiction, because $tx'(t) > 0$, and $x(t)$ is bounded for large $t$. Thus, every bounded solution of (*) is oscillatory.

*Remark.* For $n = 1$ and $g = g(x, x')$, Theorem 1 coincides with Theorem 2 in [*4*].

2. Theorem 2. *Suppose that in* (*) *we have* $n > 1$, *and moreover*:

(i) $p$ *as in Theorem* 1;

(ii) $g : \mathbb{R} \to \mathbb{R}$, $g \in \mathbf{C}'(-\infty, +\infty)$, $xg(x) > 0$ *for* $x \neq 0$, $g'(x) \geqslant 0$ *for* $|x| \in [K, +\infty)$ (*K is some positive constant*), *and*

$$\int_{\epsilon}^{+\infty} u^{1/2}du/g(u) < +\infty, \qquad \int_{-\epsilon}^{-\infty} |u|^{1/2}du/g(u) < +\infty$$

*for every* $\epsilon > 0$; *then every solution of* (*) *is oscillatory.*

*Proof.* Suppose that $x(t)$ is a solution of (*) defined on $I$. Then, if $x(t)$ is bounded on $I$, Theorem 1 implies that it must be oscillatory. Now assume that $x(t)$ is unbounded and positive for all large $t$; then there exists a $t_0^* \geqslant t_0$ such that $x(t) \geqslant K$ for every $t \in [t_0^*, +\infty)$. Here, following Mikusinski ([*6*], pp. 38–39), it suffices to distinguish two cases:

*Case* 1. There exists a $t_1 > t_0^*$ such that

$$(-1)^m x^{(2n-m)}(t) < 0 \qquad \text{for every} \quad m = 1, 2, \ldots, 2n-1 \tag{12}$$

and every $t \geqslant t_1$. By differentiation of the function

$$F_1(t) = t^{2n-1} x^{(2n-1)}(t)/g(x(t)), \tag{13}$$

we obtain

$$\begin{aligned} F_1'(t) &= -t^{2n-1}p(t) + (2n-1)\, t^{2(n-1)} x^{(2n-1)}(t)/g(x(t)) \\ &\quad - t^{2n-1} x^{(2n-1)}(t)\, x'(t)\, g'(x(t))/g(x(t))^2 \\ &\leqslant -t^{2n-1}p(t) + (2n-1)\, t^{2(n-1)} x^{(2n-1)}(t)/g(x(t)) \end{aligned} \tag{14}$$

where $g'$ means differentiation with respect to $x$. From (14), by integration from $t_1$ to $t \geqslant t_1$, we get

$$\begin{aligned} F_1(t) - F_1(t_1) - (2n-1) \int_{t_1}^{t} s^{2(n-1)} x^{(2n-1)}(s)\, ds/g(x(s)) \\ \leqslant -\int_{t_1}^{t} s^{2n-1} p(s)\, ds \to -\infty \qquad \text{as} \quad t \to +\infty, \end{aligned} \tag{15}$$

which implies

$$\lim_{t\to+\infty} \int_{t_1}^{t} s^{2(n-1)} x^{(2n-1)}(s)\, ds/g(x(s)) = +\infty. \tag{16}$$

Now, since

$$\begin{aligned} \int_{t_1}^{t} s^{2(n-1)} x^{(2n-1)}(s)\, ds/g(x(s)) &= s^{2(n-1)} x^{(2n-2)}(s)/g(x(s))\Big]_{t_1}^{t} \\ &\quad -2(n-1) \int_{t_1}^{t} s^{2n-3} x^{(2n-2)}(s)\, ds/g(x(s)) \\ &\quad + \int_{t_1}^{t} s^{2(n-1)} x^{(2n-2)}(s)\, g'(x(s))\, x'(s)\, ds/g^2(x(s)), \end{aligned} \tag{17}$$

Eq. (16) yields

$$\int_{t_1}^{+\infty} t^{2n-3} x^{(2n-2)}(t)\, dt/g(x(t)) = -\infty. \tag{18}$$

Similarly by induction,

$$\begin{aligned} \int_{t_1}^{+\infty} t^{2n-(m+1)} x^{(2n-m)}(t)\, dt/g(x(t)) &= +\infty, \qquad \text{for} \quad m = 1, 3, \ldots, 2n-1 \\ \int_{t_1}^{+\infty} t^{2n-(m+1)} x^{(2n-m)}(t)\, dt/g(x(t)) &= -\infty, \qquad \text{for} \quad m = 2, 4, \ldots, 2(n-1). \end{aligned} \tag{19}$$

For $m = 2n - 1$, the first of (19) gives

$$\lim_{\tau \to +\infty} \int_{t_1}^{\tau} x'(t)\, dt/g(x(t)) = \lim_{\tau \to +\infty} \int_{x(t_1)}^{x(\tau)} du/g(u) = +\infty, \tag{20}$$

which, by hypothesis (ii) is impossible.

*Case* 2. For some index $2n - 2i$ $(2n - 2i \geqslant 2)$, there exists a $t_1 > t_0^*$ such that

$$\begin{aligned} &x^{(2n-2i)}(t) > 0, \qquad (-1)^m x^{(2n-m)}(t) < 0, \qquad \text{for} \quad m = 0, 1, 2, \dots, 2i - 1, \\ &\quad x(t) \geqslant \mu t^{2n-2i}, \end{aligned} \tag{21}$$

for every $t \geqslant t_1$ and some $\mu > 0$. Then from (19) (which still holds for $m = 2i - 1$) we obtain

$$\int_{t_1}^{+\infty} t^{2n-2i} x^{(2n-2i+1)}(t)\, dt/g(x(t)) \leqslant \lambda \int_{t_1}^{+\infty} t^{2n-2i}\, dt/g(\mu t^{2n-2i}) = +\infty, \tag{22}$$

where $\lambda$ is an upper bound for $x^{(2n-2i+1)}(t)$ which is decreasing for $t \geqslant t_1$.

Thus, since (22) is impossible (as it can be easily seen by putting $u = t^{2n-2i}$ and use of (ii)), the theorem is true.

*Note added in proof.* The integral conditions on $g$ (in Th. 2) can be replaced by the following weaker ones:

$$\int_{\epsilon}^{+\infty} du/g(u) < +\infty, \qquad \int_{-\epsilon}^{-\infty} du/g(u) < +\infty.$$

In fact, from the Taylor series expansion of $x'(t)$ (Case 2) we easily obtain $t^{2n-2i}x^{(2n-2i+1)}(t) \leqslant Mx'(t)$ ($M$ some positive constant) for all sufficiently large $t$, and this combined with (22) gives us the desired contradiction:

$$\int_{t_2}^{+\infty} t^{2n-2i} x^{(2n-2i+1)}(t)\, dt/g(x(t)) \leqslant M \int_{x(t_2)}^{+\infty} du/g(u) = +\infty.$$

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